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A NEW THEORY OF THE RULE OF SIGNS.

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Prof. De Morgan has remarked that Sturm's Theorem "is the complete *theoretical* solution of a difficulty upon which energies of every order have been employed since the time of Descartes." But the amount of labor which it often requires, has turned back the attention of Algebraists to shorter methods. Accordingly the design of the present article will be to fix upon one central principle, easy of recollection and application, strictly demonstrated, and approaching progressively toward the completeness of Sturm's Theorem by the shortest modes of inspection. A first advance of mine in the investigation of a particular case of this kind was inserted in the Mathematical Monthly for 1858.

I. In every equation, wrote Descartes, there may be as many positive roots as there are changes of sign or passages from the sign + to the sign —, or the contrary; and as many negative roots as there are successions of the same sign. In this equation for example, $x^3 - 17x^2 + 79x - 63 = 0$, there are three changes of sign, and the three roots are positive, namely 1, 7, 9. Let us multiply it by $x + 4$; we shall have this result $x^4 - 13x^3 + 11x^2 + 253x - 252 = 0$; where there are actually three changes of sign, which indicate the three positive roots, with a succession of the same sign by reason of the negative root.

The general statement, being left by Descartes incomplete and without demonstration, encountered much opposition, according to Montucla, from whose history the above extract is translated; but improvments have followed slowly and surely. In entering this field of investigation, let us first consider a new and important extension.

II. *Order of signs and magnitude of the Roots.* In multiplying the first equation from Descartes by $x + 4$, the inquiry arose in the mind of the writer, where will the new permanence enter among the three former variations? Denoting a permanence of sign by p, and a variation of sign by v, it was interesting to observe, it had located in the order $vvpv$ corresponding with the roots 9, 7, — 4, 1, arranged in descending order of magnitude.

Applying this clue to equations of the second degree, it was readily proved that the right hand variation or permanence defines the sign of the least root, and the next defines the sign of the greatest root. For illustration, the quadratic $x^2 - 4x - 21 = 0$, indicates vp; and the corresponding roots are 7, — 3, in descending magnitude, as before. Again take the biquadratic equation $x^4 + 17.658x^3 - 65.341x^2 + 36.5087x + 2.1576 = 0$.

Scale. *pvvp*, or — + + —. Now the four roots are actually, $x = -20.87$, $+ 2.50$, $+ 0.768$, $- 0.0538$. All of which correspond with the scale both in sign and in order of magnitude.

From induction of numerous resolved examples, this *double correspondence*, suggested as above described, is generally true. As such it will be presently indicated in the new Rule. In the only exception yet found, the two roots $+ .76$, $- .73$, border upon equal roots, where transposition is admissible.

Before proceeding further, let us multiply the first equation from Descartes by a factor having imaginary roots, as by $x^2 + 20x + 110$; the product will be, $x^5 + 3x^4 - 151x^3 - 353x^2 + 7430x - 6930 = 0$.

Scale. *pvpvv*, or — + — + +.

It should be observed that the two imaginary roots are accounted negative; they do not enter adjacent to each other, but are alike. If we had multiplied by $x^2 + 2x + 10$, the product would have given the scale + + + + +; here the two imaginary roots enter as positive, but still alike Therefore the factor representing two imaginary roots should generally have the form of $x^2 \pm ax + b$, where b is positive.

By way of corollary, in the case of equal roots, with or without contrary signs, the corresponding signs of the scale may be indifferently transposed, since there is no advance in value. And in the only other instance known at present, of an inverted order, the two roots $+ .764$, $- .732$ were *nearly* equal. Thus with the occasional exception of roots approximately equal, the double correspondence of the scale has so far been found entirely applicable.

III. *Demonstration of Descartes' rule.* The common method consists chiefly in proving that the introduction of one more positive root into an equation, is always accompanied by at least one more variation of sign. Thus let a denote any positive root, to be introduced into the following equation:

$$(1) \qquad x^{m-1} + Ax^{m-2} + Bx^{m-3} + \ldots Tx + U = 0.$$

Multiplying by $x - a$ to introduce the root a, we have the equation,

$$(2) \quad x^m \begin{array}{l|l|l|l|l} +\ A & x^{m-1} + \ B & x^{m-2} + \ C & x^{m-3} \ldots + \ U & x \\ -\ a & \quad -aA & \quad -aB & \quad -aT & \quad -aU = 0. \end{array}$$

When each term containing x in equation (2) has the same sign as its upper letter, the order of signs will evidently be the same as in (1), with the gain of one variation at the end, where $-aU$ has a sign contrary to that of the preceding term determined by U.

But more generally the equation (2) commences on the left, with a compartment of one or more terms having the same signs as in (1) or the upper letters; then comes a reverse compartment of terms with signs opposite to those of (1), and which consequently give the same permutations as in (1), and so on alternately; thus,

(2) Original sign or signs. Reverse. Original. Reverse. . . . Reverse.
(2) $O\ R\ O\ R\ O\ R\ O\ R\ O\ldots R.$

Let $n + 1$ denote the number of left hand junctions of the reverse with original signs, in the order $(O\ R)$, at each of which a variation is always gained; then will n evidently denote the number of right hand junctions in the order $(R\ O)$, at each of which a variation may be either lost or gained; which results are thus proved:

Let A and $(B - aA)$ respectively denote the signs of the last coefficient of an original and the first of the next reverse compartment. In order that the part or product $-\,a\times A$ may have a sign opposite to that of B, as well as exceed B, it is evident that A and B must have like signs, and so give a permanence in (1), while the corresponding terms in (2) give a variation. Thus at the left hand junction of every reverse compartment $(O\ R)$, a variation is gained.

Again let $-\,C$ and $(D - aC)$ respectively denote the signs of the last of a reverse, and the first of the next original compartment. At this right hand junction, the signs in (1) will evidently be those of C, D; the signs in (2) will be those of $-\,C$, D. If the former denote a permanence, the latter will denote a variation, and conversely.

Thus at the $n + 1$ left hand junctions of reverse compartments, $n + 1$ variations are always gained in (2) over (1), while at the n right hand junctions only n variations can be lost. Hence the introduction of a positive root is always attended by the net gain of at least one variation in (2) over the previous number in (1).

The nature of the results may also be exhibited as follows:

$$\left.\begin{array}{l}\text{At } n+1 \text{ left hand}\\ \text{junctions, } (O\ R)\ldots\end{array}\right\}\begin{array}{l}\text{in (1) } A,\ B, \text{ permanence only.}\\ \text{in (2) } A,\ -B, \text{ variation only.}\end{array}$$

$$\left.\begin{array}{l}\text{At } n \text{ right hand}\\ \text{junctions, } (R\ O)\ldots\end{array}\right\}\begin{array}{l}\text{in (1) } C,\ D, \text{ variation or permanence.}\\ \text{in (2) } -\,C,\ D, \text{ permanence or variation.}\end{array}$$

The result is entirely conclusive, that the number of real positive roots cannot exceed the variations of sign. And in like manner, the multiplication by $x + b$ to introduce a negative root, would necessarily give at least one more permanence of signs; whence a similar conclusion follows.

In the next place, let us suppose the signs of the first and last terms of equation (1) to be alike; then must the number of variations along its terms, be zero or an even number; while the first and last terms of (2) being then unlike, the number of variations in (2) must be an odd number. Again, let us suppose the first and last terms of (1) to have unlike signs, which must indicate in total, an odd number of variations; while the first and last terms of (2) being then alike, must indicate an even number in (2). Thus in every case, the gain of variations in (2) over the previous number in (1), being the difference between an odd and an even number, is always an odd number

IV. *Incomplete Equations.* We are at liberty to supply the place of deficient terms with zero terms or infinitesimals, having such signs as to render the number of variations a minimum; and so of the permanences. Now if the inserted zeros have the same sign as the preceding term, as in $+ x^5 + 0 + 0 — x^2$, they can evidently give no new variation; and their insertion is superfluous, in respect to *positive* roots.

In respect to *negative* roots, an even number of zero terms whose signs alternate with the preceding, as in $— x^5 + 0 — 0 + x^2$, can give no new permanence; since the last zero has the same sign as the preceding real term Consequently the insertion of zero terms is superfluous, when the difference of the including exponents is an *odd* number, or the number of vacant terms is even.

But when the difference is an *even* number, between the two including exponents, let us *first* suppose the two including signs to be unlike, as in $+ 3x^6 — 0 + 0 — 0 — 5x^2$. Here the alternating signs admit of no permanence except at the junction of the last zero with the next real term; which invariably gives one permanence. As the terms $+ 3x^6 — 5x^2$ have previously indicated one *variation,* and now are made by the inserted zero to indicate one permanence, the total result will be rendered by simply copying both the given signs + and —, into the scale of roots, as described in the new Rule.

Lastly, the difference of the two including exponents being still an *even* number, let us *secondly* suppose the signs of the two including terms to be alike, as in $+ x^4 — 0 + 2x^2$. In this, and all such cases, the alternate signs admit not of any permanence. And the previous like signs gave no variation. Hence the two corresponding roots are not real but imaginary; and the correct result will be rendered by temporarily omitting either of the like signs, or by remembering to regard the succession in $+ x^4 + 2x^2$, and such terms, as a permanence of imaginary roots only, and omitting it from the scale of the new Rule.

Thus the effect of zero or vacant terms can be generalized into the Rule, so that their further insertion is no longer needed. .Nor does it appear necessary here to give separate precepts for the corresponding imaginary roots; since the number of signs or roots of the scale, positive and negative, subtracted from the degree of the equation will give the even excess of imaginary roots, beyond any pair or pairs of real roots noted in the scale.

The invention ot Fluxions or the Calculus has opened the way to the next simple criterion of imaginary roots; to which let us now proceed.

V. *The Quadratic Type.* This name is here suggested for any three terms of an equation, where the three exponents are in the series of natural numbers, and the first and third terms, or extremes, have like signs; whatever may be the sign of the mean or middle term; such as, — $13x^2 + 10x - 49$, or $Px^5 \pm Qx^4 + Rx^3$.

The leading idea is, that, when the final differential coefficient, placed equal to zero, has imaginay roots, the original equation must have at least an equal number of them. Let it here be observed, that the types to give independent indications of imaginary roots, can only be contiguous; that is, the middle term of one must not occur in the next, but the last term of one type may be taken as the first of the next type. Let it also be remembered, to guard against a common mistake, that if all the types in an aquation give real roots, we cannot infer conversely that the roots of the original equation are all real; for imaginary roots are still possible, with certain exceptions.

Let the original equation be denoted by,

$$u = Ax^m + \ldots + Px^n + Qx^{n-1} + Rx^{n-2} + \ldots = 0.$$

Differentiating $n - 2$ times, we then substitute the reciprocal of y in place of x. Multiplying by y^{m-n+2}, and differentiating $m - n$ times with respect to y, then restoring the value of y, and omitting common factors, we obtain the isolated type,

$$n(n-1)Px^2 + 2(n-1)(m-n+1)Qx + (m-n+2)(m-n+1)R = 0.$$

Resolving this quadratic,

$$x = \frac{m-n+1}{nP}\left[-Q \pm \sqrt{(Q^2 - fPR)}\right],$$

where $$f = \frac{(m-n+2)n}{(m-n+1)(n-1)}.$$

If the quantity under the radical sign be negative, the two roots of the last derivative are imaginary, and consequently two roots of the original equation are imaginary. This is rendered very evident by tracing the curve of any equation and its derivatives, after the manner of Rolle, of which some elegant examples are given in Montucla.

It will be noticed that the value of f is the ratio of two consecutive factors of the general binomial coefficient. When the degree of the equation m is an odd number, the middle or minimum value of f is $\frac{m+3}{m-1}$, being always greater than unity. And when m is an even number, the middle or minimum value of f is $\left(\frac{m+2}{m}\right)^2$. Equi-distant from the middle, the values of f are equal to each other. Thus, When the quadratic type begins with the first term, or ends with the last term of an equation, we have $n = m$, and $f = \frac{2m}{1(m-1)}$.

When the type begins with the regular second term, or ends with the usual last term but one, then will $n = m - 1$, and $f = \frac{3(m-1)}{2(m-2)}$.

When the type begins with the normal third term, counted in from either extremity of an equation, $n = m - 2$, and $f = \frac{4(m-2)}{3(m-3)}$.

To recapitulate,—In the common *quadratic* equation two imaginary roots exist when $Q^2 - 4PR$ is negative.

In any equation of the *third* degree, two roots are imaginary, if $Q^2 - 3PR$ is negative.

In any equation of the *fourth* degree, the negative type of two imaginary roots, is $Q^2 - \frac{8}{3}PR$; that is, if P denotes the first term or R the last term of the equation. And in the middle part the type is $Q^2 - \frac{9}{4}PR$.

In any equation of the *fifth* degree the values of f are, $f = \frac{5}{2}, 2, \frac{5}{2}$. Yet the general formula may be preferable; thus, *to find f*;—

The highest exponent of the type is one factor of f, and the difference between the lowest exponent of the type and the degree of the whole equation, is the other factor of the numerator. Subtracting unity from each, gives the two factors of the denominator.

The quadratic type occurs, for example, in the last three terms of,

$$x^4 - x^3 - 13x^2 + 16x - 48 = 0.$$

Here $f = \frac{8}{3}$, and $Q^2 - fPR = (16)^2 - \frac{8}{3} \times 13 \times 48 = -1408$. This result being negative, it follows that two roots are imaginary.

The quadratic type occurs in the first three terms of the equation,

$$x^3 + 3x^2 + 7x + 4 = 0.$$

Here $3^2 - 3 \times 1 \times 7$ being negative, two roots are imaginary.

It may be remarked that if the first and last terms of the type had unlike signs, no calculation were needed to show that since $-PR$ is then positive, the criterion would be positive and indecisive.

VI. *The Cubic Type.* By this term we designate any three or four terms of an equation, which terms divided by their lowest power of x would give the form of the common cubic equation. For example,

$$+ Px^n + Qx^{n-1} + Rx^{n-2} + Sx^{n-3}.$$

Let us suppose the given equation to be first divided by P, and let $Q \div P = Q'$, etc., then,

$$A'x^m + \ldots + x^n + Q'x^{n-1} + R'x^{n-2} + S'x^{n-3} + \ldots = 0.$$

Differentiating $n - 3$ times, and dividing by $(n-3)\ldots 3.2.1$,

$$\left(\ldots\right)x^{m-n+3}\ldots+\frac{n(n-1)(n-2)}{1.2.3}x^3+\frac{(n-1)(n-2)}{1.2}Q'x^2+\frac{n-2}{1}R'x$$
$$+S'=0.$$

Dividing through by x^{m-n+3}, then substituting y in place of the reciprocal of x, differentiating $m-n$ times with respect to y, then restoring x and dividing by the left hand coefficient,

$$x^3+\frac{3(m-n+1)}{n}Q'x^2+\frac{3(m-n+1)(m-n+2)}{n(n-1)}R'x$$
$$+\frac{(m-n+1)\ldots(m-n+3)}{n(n-1)(n-2)}S'=0.$$

If this final derivative has two imaginary roots, to be ascertained presently by solution or by Sturm's theorem, then must the preceding derivative, and consequently the original equation, have two such roots, as previously described. Causing the second term to disappear by making

$$x=\frac{(m-n+1)}{n}\left(z-Q'\right),$$

$$z^3+3(fR'-Q'^2)z+2Q'^3-3fQ'R'+f\frac{(m-n+3)n}{(m-n+1)(n-2)}.S'=0.$$

The general equation $z^3+qz+r=0$, is proved in Algebra to have two imaginary roots whenever $-(\frac{1}{3}q)^3-(\frac{1}{2}r)^2$ is negative. Substituting and dividing by Q'^6, which cannot alter the sign, the criterion becomes,

$$\left(1-\frac{fR'}{Q'^2}\right)^3-\left(1-\frac{3fR'}{2Q'^2}+\frac{f(m-n+3)n}{2(m-n+1)(n-2)}\,\frac{S'}{Q'^3}\right)^2=neg.$$

Or multiplying and dividing the last term by $(fR'\div Q'^2)^2$ denoted by f'^2, and introducing g' to be presently described, the criterion is,

$$(1-f')^3-(1-\tfrac{3}{2}f'+\tfrac{1}{2}f'^2g')^2=negative.$$

Expanding and multiplying by $4\div f'^2$, which cannot alter the sign, the criterion becomes symmetrical; thus,

$$(C)\qquad 3-4(f'+g')+6f'g'-(f'g')^2=neg.$$

Now to facilitate the practical application, let us suppose this criterion to be the ordinate, and g' the abscissa of a continuous curve. When g' is infinite positive or negative, the criterion is evidently negative in both branches of the curve; while the intermediate portion between the two roots of g', is positive. Thus when the criterion is placed equal to zero, the two roots g_1, g_2 will be limits; outside of, and not between which, the actual value of g' must fall to indicate two imaginary roots. We find,

$$g_1=\frac{3f'-2-2(1-f')^{\frac{3}{2}}}{f'^2}.$$

$$g_2 = \frac{3f' - 2 + 2(1 - f')^{3}}{f'^2},$$
$$= \tfrac{3}{4} + \tfrac{1}{8}f' + \tfrac{3}{64}f'^2 + \tfrac{3}{128}f'^3 + \ldots .$$

These formulas will next be tabulated; the degree of the given equation being m; the cubic type, $Px^n + Qx^{n-1} + Rx^{n-2} + Sx^{n-3}$,

$$f' = \frac{n(m-n+2)}{(n-1)(m-n+1)} \cdot \frac{PR}{Q^2}, \qquad g' = \frac{n-1}{n-2} \cdot \frac{m-n+3}{m-n+2} \cdot \frac{QS}{R^2}.$$

Here the first factors are formed alike from the exponents of P, R and Q, S, by the simple precept before given under the quadratic type.

OUTLINE TABLE OF THE LIMITS.

f'	g_1	g_2	f'	g_1	g_2	f'	g_1	g_2
+1.00*	+1.000	+1.000	0.0	—∞	+0.750	—2.000	—4.598	+0.598
0.90	+0.786	0.942	—0.1	—460.7	0.738	—3.000	—3.000	0.556
0.80	+0.345	0.904	—0.2	—130.7	0.726	—5.000	—1.855	0.496
0.75	0.000	0.889	—0.3	—65.16	0.716	—7.000	—1.385	0.446
0.70	—0.467	0.875	—0.4	—40.71	0.706	—9.000	—1.139	0.423
0.60	—1.961	0.850	—0.5	—28.70	0.697	—20.00	—0.636	0.326
0.50	—4.828	0.828	—0.6	—21.80	0.688	—40.00	—0.404	0.252
0.40	—10.81	0.810	—0.7	—17.41	0.680	—60.00	—0.315	0.214
0.30	—25.24	0.793	—0,8	—14.42	0.672	—80.00	—0.266	0.190
0.20	—70.78	0.778	—0.9	—12.27	0.664	—100.0	—0.233	0.173
+0.10	—340.7	+0.766	—1.0	—10.66	+0.657	—1000.	—0.066	+0.060

In respect to vacant terms, if either P or S is zero then f' or g' is zero; and by the table, the other g' or f' must equal or exceed 0.75 to indicate two imaginary roots. That is, simply, $0.75 - f' =$ *negative.*

When $Q = 0$; we multiply the criterion by Q^2R^2 and then make $Q = 0$, reducing the criterion to $-4PR^3 - f(gPS)^2$. When P and R have *like* signs, this is negative, so that two roots are imaginary, as under the quadratic type. When P and R have *unlike* signs, $-PR$ is positive, and the criterion is,

$$(C') \qquad -\frac{R^3}{P} - \frac{fg^2}{4} . S^2 = \textit{neg.}, \text{ where } g = \frac{n-1}{n-2} \cdot \frac{m-n+3}{m-n+2}.$$

In like manner, when $R = 0$, the criterion is,

$$(C'') \qquad -\frac{Q^3}{S} - \frac{gf^2}{4} . P^2 = \textit{negative.}$$

(*To be continued.*)

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*This first value indicates generally two imaginary roots, otherwise two equal roots.